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EPT graphs on bounded degree trees

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Abstract

An undirected graph G is called an **EPT graph** if it is the edge intersection graph of a family of paths in a tree. In this paper, we answer negatively the question posed by Golumbic et al. [4]: Can any EPT graph without induced cycles of size greater than h be represented in a host tree with maximum degree h?

1 Introduction and previous results

A graph G is called an **EPT graph** if it is the edge intersection graph of a family of paths in a tree. An **EPT representation** of G is a pair $\langle \mathcal{P}, T \rangle$ where $\mathcal{P}$ is a family $(P_v)_{v \in V(G)}$ of subpaths of the host tree T satisfying that two vertices v and v' of G are adjacent if and only if P_v and $P_{v'}$ have at least two vertices (one edge) in common. When the maximum degree of the host tree T is h, the EPT representation of G is called an $(h, 2, 2)$-representation of G. The class of graphs which admit an $(h, 2, 2)$-representation is denoted by **[h,2,2]**. Notice that the class of EPT graphs is the union of the classes $[h, 2, 2]$ for $h \geq 2$. In [3] it is proved that the

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The EPT graphs are used in network applications, where the problem of scheduling undirected calls in a tree network is equivalent to the problem of coloring an EPT graph (see [2]). The communication network is represented as an undirected interconnection graph, where each edge is associated with a physical link between two nodes. An undirected call is a path in the network. When the network is a tree, this model is clearly an EPT representation. Coloring the EPT graph, such that two adjacent vertices have different colors, implies that paths sharing at least one common edge in the EPT representation have different colors, meaning that undirected calls that share a physical link are scheduled in different times.

Definition 1.1. *Let $\langle \mathcal{P}, T\rangle$ be an EPT representation of a graph G. A* ***pie of size n*** *is a star subgraph of T with central vertex q and neighbors q_1,...,q_n such that each "slice" $q_i\, q\, q_{i+1}$ for $1 \leq i \leq n$ is contained in a different member of $\mathcal{P}$; addition is assumed to be module n. (See Figure 1).*

Let $\langle \mathcal{P}, T\rangle$ be an EPT representation of a graph G. It was proved (see [4]) that if G contains a chordless cycle of length $n \geq 4$, then $\langle \mathcal{P}, T\rangle$ contains a pie of size n.

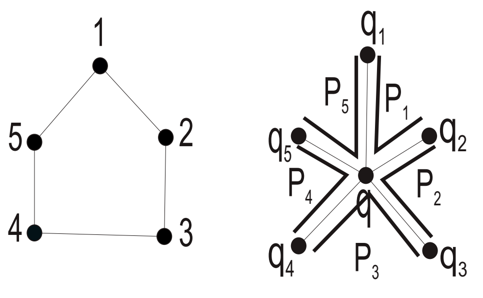

Figure 1: The cycle C_5 and an EPT representation: a pie of size 5.

Notice that, in a pie, the paths corresponding to k consecutive vertices of the cycle cover $k+1$ edges incident in q.

Since a pie for a cycle C_n of size n requires a host tree with a vertex of degree n, we have the following theorem.

Theorem 1.1. [3] If $G \in [h, 2, 2]$, then G is $\{C_n : n > h\}$-free.

In [4], Golumbic, Lipshteyn and Stern study several aspects of the classes $[h, 2, 2]$ for $h \geq 2$. In particular, they proved that, in the class of EPT graphs, the reciprocal of Theorem 1.1 is true for $h = 3$ and false for $h = 4$. They used general results about Chordal graphs and Weakly Chordal graphs (see [1] for definitions) which apply only when $h = 3$ or $h = 4$. Their proof is based in the facts that $[3, 2, 2] = EPT \cap Chordal$ and $[4, 2, 2] = EPT \cap Weakly\ Chordal$ [5]. They let open the question about whether the converse of Theorem 1.1 is true for $h \geq 5$. In this paper, we answer negatively this question. We succeed in describing, for every $h \geq 5$, an EPT graph that is $\{C_n : n > h\}$-free and, however, it can not be represented in a host tree with maximum degree h.

2 Our results

In what follows we depict the graphs which will be used to answer the question posed by Golumbic et al. [4]: Can any EPT graph without induced cycles of size greater than h be represented in a host tree with maximum degree h?

Definition 2.1. *Let n_1, n_2 and n_3 be positive integers. A* ***general prism*** *F_{n_1,n_2,n_3} consists of two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$; and three disjoint chordless paths Q_i for $1 \leq i \leq 3$ of length (number of edges) n_i and extreme vertices a_i, b_i respectively. (See Figure 2).*

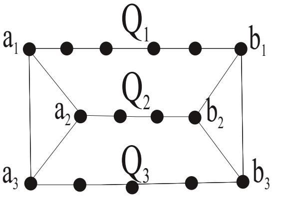

Figure 2: The general prism $F_{5,3,4}$.

Observe that:

(1) $F_{1,1,1}$ is isomorphic to the complement of the graph C_6.

(2) By symmetry, in a general prism, the order of the subindexes is irrelevant, that is, $F_{n_1,n_2,n_3} \simeq F_{n_1,n_3,n_2} \simeq F_{n_2,n_1,n_3} \simeq F_{n_2,n_3,n_1} \simeq F_{n_3,n_2,n_1} \simeq F_{n_3,n_1,n_2}$.

(3) The general prism F_{n_1,n_2,n_3} has $n_1 + n_2 + n_3 + 3$ vertices.

(4) The general prism F_{n_1,n_2,n_3} without the vertices a_1 and b_3 is a chordless path of lenght $n_1 + n_2 + n_3$.

(5) The general prism F_{n_1,n_2,n_3} contains exactly three induced cycles:

$$\begin{array}{lll} Q_1\widehat{Q_2} & \text{of lenght} & n_1 + n_2 + 2; \\ Q_1\widehat{Q_3} & \text{of lenght} & n_1 + n_3 + 2; \\ Q_2\widehat{Q_3} & \text{of lenght} & n_2 + n_3 + 2; \end{array}$$

where $Q_i\widehat{Q_j}$ is the concatenation of the paths Q_i and the reverse of Q_j.

Lemma 2.1. The general prism F_{n_1,n_2,n_3} is an $[h, 2, 2]$ graph for $h = n_1 + n_2 + n_3 + 2$.

Proof: Let T be a star with central vertex q and pendant vertices q_i for $1 \leq i \leq n_1+n_2+n_3+2$. For each i, $1 \leq i \leq n_1+n_2+n_3+1$, let P_i be the subpath of the host tree T induced by the vertices q_i, q and q_{i+1}. The edge intersection graph of this family of paths is a chordless path isomorphic to F_{n_1,n_2,n_3} without the vertices a_1 and b_3; thus we need to add to this family two other paths, say P and P', representing the vertices a_1 and b_3 respectively.

Let P be the subpath of T induced by the vertices q_1, q and $q_{n_1+n_2+2}$; and let P' be the subpath of T induced by the vertices $q_{n_1+n_2+n_3+2}$, q and q_{n_1+1} (See Figure 3). It is clear that the vertex represented by P is adjacent to the ones represented by P_1, $P_{n_1+n_2+1}$ and $P_{n_1+n_2+2}$; while the vertex represented by P' is adjacent to the ones represented by $P_{n_1+n_2+n_3+1}$, P_{n_1} and P_{n_1+1}; thus the proof follows. $\square$

The following theorem shows that induced cycles are not the only structures in an EPT graph which impose conditions on the vertex degrees of the host tree.

Theorem 2.1. Let $h = n_1 + n_2 + n_3 + 1$. The general prism F_{n_1,n_2,n_3} is not an $[h, 2, 2]$ graph, however it is $\{C_n : n > h\}$-free.

Proof: The case $n_1 = n_2 = n_3 = 1$ was solved by Golumbic et al. in [4] as we have mentioned in Section 1. Thus, we assume $n_1 \geq 2$, which means that Q_1 has at least three vertices.

Suppose $F_{n_1,n_2,n_3} \in [h, 2, 2]$ for $h = n_1 + n_2 + n_3 + 1$; and let $\langle \mathcal{P}, T \rangle$ be an EPT representation satisfying that the degree of every vertex of T is less or equal than h. Since $Q_1\widehat{Q_2}$ induce a cycle of size $n_1 + n_2 + 2$, in the representation there is a pie of size $n_1 + n_2 + 2$. Call q to the central vertex of this pie.

On the other hand, since $Q_1\widehat{Q_3}$ also induces a cycle, in the representation there is another pie. We claim that the central vertex of this second pie is the same vertex q. Indeed, it follows from:

(i) in a pie, the vertex intersection of 3 paths corresponding to 3 consecutive vertices is exactly the central vertex of the pie;

(ii) the three first vertices of Q_1 are consecutive in both cycles $Q_1\widehat{Q_2}$ and $Q_1\widehat{Q_3}$.

Now, the paths corresponding to the vertices of $Q_1\widehat{Q_2}$ cover $n_1 + n_2 + 2$ edges of T incident in q. On the other hand, the paths corresponding to the $n_3 - 1$ internal vertices of Q_3 must cover n_3 edges of T incident in q which are different from the $n_1 + n_2 + 2$ covered by the vertices of $Q_1\widehat{Q_2}$, because no internal vertex of Q_3 is adjacent to a vertex of $Q_1\widehat{Q_2}$.

We obtain that q has degree at least $n_3 + (n_1 + n_2 + 2)$, which contradicts the assumption that every vertex of T has degree at most $h = n_1 + n_2 + n_3 + 1$.

Finally, by fact (5) and since $n_1 + n_2 + 2 \leq n_1 + n_2 + n_3 + 1$; $n_1 + n_3 + 2 \leq n_1 + n_2 + n_3 + 1$ and $n_2 + n_3 + 2 \leq n_1 + n_2 + n_3 + 1$, the general prism F_{n_1,n_2,n_3} has no cycles of size greater than $h = n_1 + n_2 + n_3 + 1$. $\square$

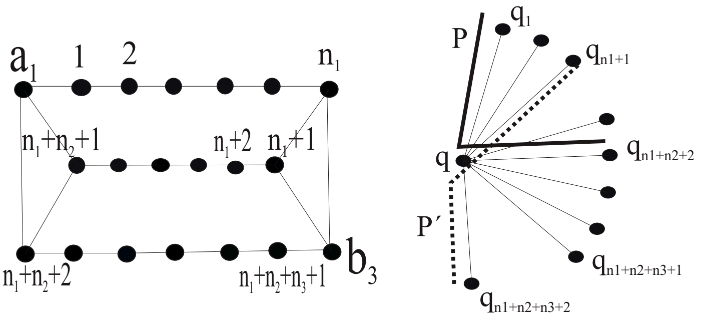

Figure 3: F_{n_1,n_2,n_3} and an $(h,2,2)$-representation, where $h = n_1 + n_2 + n_3 + 2$.

Finally, we give the following theorem which generalizes Theorem 1.1:

Theorem 2.2. Let $h \geq 4$. If $G \in [h,2,2]$, then G is $\{F_{n_1,n_2,n_3}, C_n\colon\ n > h$ and $n_1 + n_2 + n_3 + 1 = h\}$-free.

We are working on finding the complete list of EPT graphs which are minimal forbidden induced subgraphs for the class $[h,2,2]$.

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